Transverse Killing and twistor spinors associated to the basic Dirac operators

A.M. Ionescu, V. Slesar, M. Visinescu and G.E. Vîlcu

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Abstract

We study the interplay between basic Dirac operator and transverse Killing and twistor spinors. In order to obtain results for general Riemannian foliations with bundle-like metric we consider transverse Killing spinors that appear as natural extension of the harmonic spinors associated with the basic Dirac operator. In the case of foliations with basic-harmonic mean curvature it turns out that these Killing spinors and twistor spinors coincide with the standard definition. We obtain the corresponding version of classical results on closed Riemannian manifold with spin structure, extending some previous results.

Keywords:*Riemannian foliations; basic Dirac operator; transversal Killing spinor; transversal twistor spinor.*

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1 Introduction

On differentiable closed Riemannian manifolds the interplay between the spectrum of associated natural differential operators such as Dirac operator and its square, the Laplace operator- on a side, and the geometry of the underlying manifold- on the other side, has already become a classical research subject.

If we consider a foliated structure on our manifold such that the metric tensor field is bundle-like (i.e. the manifold can be locally described as a Riemannian submersion [28]), then similar operators can be defined in this particular setting, standing as important tools for the study of the transverse spectral geometry of the foliations [31]. Furthermore, on the foliated manifold can be considered the existence of a symplectic structure or a CR-submanifold (see e.g. [4, 6, 20, 32, 33]) which is also known to interact with the canonical differential operators [27].

Concerning the Dirac-type operators, the *transversal Dirac* operator for Riemannian foliations was introduced in [12]. In the particular case of a Riemannian foliation with basic mean curvature form, this operator is used to defined the *basic Dirac* operator, which is a symmetric, essentially self-adjoint and transversally elliptic [12]. As pointed out by the authors, Dirac-type operators defined

in this particular framework are relevant at least in $\mathbb{R}^4$, when the Yang-Mills equations can be dimensionally reduced, yielding magnetic monopole equations.

In general the procedure of reducing a dynamical system to two or more systems of lower dimensions with respect to foliations of the manifold of which the dynamics take place has proved to be an useful tool in order to relate different integrable systems together with their associated symmetries. Sometimes the reverse of the reduction procedure is used to investigate difficult dynamical systems. It is also possible a sort of *unfolding* of the initial dynamics by imbedding it in a larger one which is easier to integrate and then projecting the solution back to the initial manifold [22].

On the other side, concerning the relevance of foliated manifolds, among a number of possible applications we mention the problem of incorporating physical chiral spinors in four dimensions in the framework of $N = 1, D = 11$ model of supergravity [35]. In this respect, a G-foliated model of the 7-dimensional internal manifold resulting from $N = 1, D = 11$ model of supergravity was proposed in [5]. The ground-state compactification of the 11-dimensional manifold M_{11} into $M_7 \times M_4$ is the result of a global submersion M_{11} into M_4, giving rise to a foliation of dimension 7 and codimension 4.

Now, regarding the spectral properties of a Dirac operator in the standard framework of a closed, Riemannian manifold we refer to [3, 8, 15]; in this setting the so called Killing spinors are known to be related to the spectrum of Dirac operator in a particular way; beside the fact that they are a tool which allow us to construct Killing vector fields (i.e. the corresponding flow is represented by local isometries) the Killing spinors are exactly the eigenspinors corresponding to the lower eigenvalue of Dirac operator [3, 15], so the geometrical conditions that insure the existence of Killing spinors and the realization of the limiting case for the Dirac spectrum are the same. A generalization of the Killing spinors is represented by twistor spinors [3, 11].

On Riemannian foliations transversal Killing and twistor spinors were defined and studied in [10, 17, 18].

The main goal of this paper is to obtain the corresponding interplay between transversal Killing spinors and basic Dirac operator in the particular setting of Riemannian foliations. In order to made such an achievement in the general setting of Riemannian foliations we define transversal Killing spinors as natural extension of the harmonic spinors associated with the basic Dirac operator (see Section 2). We also extend in a natural way the previously defined twistor spinors [18].

For the particular case of Riemannian foliation with basic-harmonic curvature, which is the most convenient setting, this definition coincide with the previous definition [10, 17, 18], and our results turn out to be generalization of [17, 18]. Moreover, in the standard manner [12, 17], the absolute case (when the manifold is foliated by points) becomes a generalization of the case of closed Riemannian manifolds.

In the second section we introduce the main geometrical object we are dealing with. In the third section we derive the main results and point out the specific features of the above Killing spinors, while in the fourth section of the paper we

study twistor spinors in the presence of a foliated structure. In the final part of the paper we point out some possible applications of the results and some physical considerations.

2 Geometric objects related to the transverse geometry of Riemannian foliations

Let us consider in what follows a smooth, closed Riemannian manifold $(M, g, \mathcal{F})$ endowed with a foliation $\mathcal{F}$ such that the metric g is bundle-like [28]; the dimension of M will be denoted by n. We also denote by $T\mathcal{F}$ the leafwise distribution tangent to leaves, while $Q = T\mathcal{F}^{\perp} \simeq TM/T\mathcal{F}$ will be the transversal distribution. Let us assume $\dim T\mathcal{F} = p$, $\dim Q = q$, $p + q = n$.

As a consequence, the tangent and the cotangent vector bundles associated with M split as follows

$$\begin{aligned} TM &= Q \oplus T\mathcal{F}, \\ TM^* &= Q^* \oplus T\mathcal{F}^*. \end{aligned}$$

The canonical projection operators will be denoted by π_Q and $\pi_{T\mathcal{F}}$, respectively.

Throughout this paper we will use local vector fields $\{e_i, f_a\}$ defined on a neighborhood of a point $x \in M$ inducing an orthonormal basis at any point where they are defined, $\{e_i\}_{1 \leq i \leq q}$ spanning the distribution Q and $\{f_a\}_{1 \leq a \leq p}$ spanning the distribution $T\mathcal{F}$.

For the study of the basic geometry of our Riemannian foliated manifold, a convenient metric and torsion-free linear connection is the so called *Bott connection* (see e.g. [31]). If we denote by ∇^g the canonical Levi-Civita connection, then on the transversal distribution Q we can define the connection ∇ by the following relations

$$\begin{cases} \nabla_U X := \pi_Q\left([U, X]\right), \\ \nabla_Y X := \pi_Q\left(\nabla^g_Y X\right), \end{cases}$$

for any smooth sections $U \in \Gamma(T\mathcal{F})$, X, $Y \in \Gamma(Q)$. In particular we can associate to ∇ the transversal Ricci curvature Ric^{∇} and the transversal scalar curvature Scal^{∇}.

We restrict the classical de Rham complex of differential forms $\Omega(M)$ to the complex of basic differential forms, defined as

$$\Omega_b(M) := \{\omega \in \Omega(M) \mid \iota_U \omega = \mathcal{L}_U \omega = 0\},$$

where U is again an arbitrary leafwise vector field, $\mathcal{L}$ being the Lie derivative along U, while ι stands for interior product. Considering now the de Rham exterior derivative d, it is possible to define the basic operator $d_b := d_{|\Omega_b(M)}$, as well as the adjoint operator, namely the *basic co-derivative* δ_b (see e.g. [1]). Let us notice that basic de Rham complex was defined independent of the metric structure g.

One differential form which is not necessarily basic is the *mean curvature form*. In order to define it, we first of all set $k^\sharp := \pi_Q \left(\sum_a \nabla^g_{f_a} f_a \right)$ to be the *mean curvature vector field* associated with the distribution $T\mathcal{F}$, while k will be the mean curvature form which is subject to the condition $k(U) = \langle k^\sharp, U \rangle$, for any vector field U, $\sharp$ being the *musical* isomorphism and $\langle \cdot, \cdot \rangle$ the scalar product in TM.

By Theorem 2.1 in [1], we have the direct sum decomposition

$$\Omega(M) = \Omega_b(M) \bigoplus \Omega_b(M)^\perp,$$

with respect to the $C^\infty-$Frechet topology. So, on any Riemannian foliation the mean curvature form can be decomposed as the sum

$$k = k_b + k_o,$$

where $k_b \in \Omega_b(M)$ is the *basic* component of the mean curvature, k_o being the orthogonal complement. From now on we denote $\tau := k_b^\sharp$.

Remark 1 *The above co-derivative operator δ_b can be written using the vector field τ and the Bott connection (see [1, 31])*

$$\delta_b = -\sum_i \iota_{e_i} \nabla_{e_i} + \iota_\tau.$$

Using the above notations, at any point x on M we consider the Clifford algebra $Cl(Q_x)$ which, with respect to the orthonormal basis $\{e_i\}$ is generated by 1 and the vectors $\{e_i\}$ over the complex field, being subject to the relations $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{i,j}$, $1 \leq i, j \leq q$, where dot stands for Clifford multiplication. The resulting bundle $Cl(Q)$ of Clifford algebras will be called the *Clifford bundle* over M, associated with Q.

We assume that the foliation $\mathcal{F}$ is transversally oriented and has a transverse spin structure. This means that there exists a principal $Spin(q)$-bundle $\tilde{P}$ which is a double sheeted covering of the transversal principal $SO(q)$-bundle of oriented orthonormal frames P, such that the restriction to each fiber induces the covering projection $Spin(q) \to SO(q)$; such a foliation is called *spin foliation* (see e.g. [13]). Similar to the standard case [29], if we denote by Δ_q the spin irreducible representation associated with Q, then one can construct the *foliated spinor bundle* $S := \tilde{P} \times_{Spin(q)} \Delta_q$. The hermitian metric on S is now induced from the transverse metric.

We have a smooth bundle action

$$\Gamma(Cl(Q)) \otimes \Gamma(S) \longrightarrow \Gamma(E),$$

denoted also with Clifford multiplication such that

$$(u \cdot v) \cdot s = u \cdot (v \cdot s),$$

for $u, v \in \Gamma(Cl(Q))$, $s \in \Gamma(S)$.

As a result S becomes a bundle of Clifford modules (see e.g. [29]).

Remark 2 *The above transverse Clifford action is obtained from the standard case of the tangent bundle as a restriction of the Clifford action from TM to the distribution L_1 (for the particular case of vector bundles see also [34]).*

The lifting of the Riemannian connection on P can be used to introduce canonically a connection on S, which will be denoted also by ∇. The *compatibility* between the Clifford action and the connection ∇ is expressed in the relation

$$\nabla_U (u \cdot s) = (\nabla_U u) \cdot s + u \cdot \nabla_U s,$$

for any $U \in \Gamma(TM)$, $u \in \Gamma(Cl(Q))$, $s \in \Gamma(S)$, extending canonically the connection ∇ to $\Gamma(Cl(Q))$.

The hermitian structure on S is now induced from the transverse metric; if we denote it by $(\cdot \mid \cdot)$, we have that $(X \cdot s_1 \mid s_2) = -(s_1 \mid X \cdot s_2)$, for any $X \in \Gamma(Q)$, s_1, $s_2 \in \Gamma(S)$.

In accordance with [12], we introduce now the *transversal Dirac operator*,

$$D_{tr} := \sum_i e_i \cdot \nabla_{e_i},$$

and its restriction to the *basic spinors* or holonomy invariant sections

$$\Gamma_b(S) := \{s \in \Gamma(S) \mid \nabla_U s = 0, \text{ for any } U \in \Gamma(T\mathcal{F})\}, \tag{1}$$

the *basic Dirac operator* which is defined using the basic component of the mean curvature form [13, 12]

$$D_b := \sum_i e_i \cdot \nabla_{e_i} - \frac{1}{2}\tau. \tag{2}$$

Remark 3 *The basic Dirac operator is transversally elliptic and essentially self-adjoint with respect to the inner product canonically associated with the closed Riemannian manifold. We emphasize the fact that the spectrum $\sigma(D_b)$ of this operator is discrete [12].*

Another way to construct the basic Dirac operator will be described in the following. We start out by modifying the Bott connection [30].

Definition 1 *The* modified connection *on the space of basic sections $\Gamma_b(S)$ is given by*

$$\bar{\nabla}_X s := \nabla_X s - \frac{1}{2}\langle X, \tau\rangle s, \tag{3}$$

for any $X \in \Gamma(TM)$ and $s \in \Gamma(S)$.

A first feature of $\bar{\nabla}$ is that it can formally replace the Levy-Civita connection in the standard definition of a Dirac operator in order to obtain the basic Dirac

operator [20]:

$$\begin{aligned}\sum_i e_i \cdot \bar{\nabla}_{e_i} &= \sum_i e_i \cdot \left(\nabla_{e_i} - \frac{1}{2} \langle e_i, \tau \rangle \right) \\ &= \sum_a e_i \cdot \nabla_{e_i} - \frac{1}{2}\tau \\ &= D_b.\end{aligned}$$

Remark 4 *It is easy to see that the modified connection on S is also compatible with the Bott connection, i.e.*

$$\bar{\nabla}_X (Y \cdot s) = \nabla_X Y \cdot s + Y \cdot \bar{\nabla}_X s,$$

for any X, $Y \in \Gamma(Q)$. However, we must observe that the modified connection does not share other classical properties, as it is not a metric connection, so the standard formal computation concerning the Levi-Civita connection on (closed) Riemannian manifold is not possible.

Now, as we have already defined the basic Dirac operator, we introduce a category of spinors intimately related to this differential operator. In order to consolidate the motivation let us study the following example.

We consider the torus $T^2 := R^2/Z^2$ with the metric $g = e^{2f(y)}dx^2 + dy^2$, for some periodic function f (see e.g. [13]). As a consequence, $\left\{\partial_y, \frac{\partial_x}{e^{f(y)}}\right\}$ represent an orthonormal basis at any point, $Q = span\{\partial_y\}$, $T\mathcal{F} = span\{\partial_x/e^{f(y)}\}$. Let us consider the trivial spin structure on the transversal circles of the foliation; the transverse Clifford multiplication by ∂_y will be represented by multiplication with the purely imaginary unit i. Using the Koszul relations, we get

$$\begin{aligned}\left\langle \nabla^g_{\frac{\partial_x}{e^{f(y)}}} \frac{\partial_x}{e^{f(y)}}, \partial y \right\rangle &= \frac{1}{2}\left(\left\langle \left[\partial y, \frac{\partial_x}{e^{f(y)}} \right], \frac{\partial_x}{e^{f(y)}} \right\rangle + \left\langle \left[\partial y, \frac{\partial_x}{e^{f(y)}} \right], \frac{\partial_x}{e^{f(y)}} \right\rangle \right) \\ &= -f'(y).\end{aligned}$$

From here we get $k = -f'(y)dy$, and using the fact that k is obviously basic, as it does not depend on x, we obtain

$$\tau = -f'(y)\partial_y.$$

On the other side it is interesting to note that

$$\begin{aligned}\delta_b k &= \left(-\iota_{\partial y} \nabla_{\partial y} + \iota_{-f'(y)\partial_y} \right) (-f'(y)dy) \\ &= f''(y) + (f'(y))^2,\end{aligned}$$

so $\delta_b k$ does not necessarily vanish, i.e. k is not basic-harmonic differential 1-form.

Finally, we calculate the basic Dirac operator

$$\begin{aligned}D_b &= i\partial_y - \frac{1}{2}(-f'(y))\, i \\ &= i\left(\partial_y + \frac{1}{2} f'(y) \right).\end{aligned}$$

Now, if we investigate the harmonic spinors, it is easy to see that the solutions of the equation $D_b s_1 = 0$ have the form $s_1 = ce^{-\frac{1}{2}f(y)}$, $c \in \mathbb{C}$.

Remark 5 *From the above calculations we see that* $\bar{\nabla}_{\partial_y} s_1 = 0$ *and* $\nabla_{\partial_y} s_1 \neq 0$. *So, even for the above simple example of Riemannian foliation we see that the harmonic spinors of the basic Dirac operator are parallel spinors with respect to the above modified connection but not with respect to the Bott connection.*

In the classical setting, a category of spinors naturally related to harmonic spinors and eigenspinors is represented by Killing spinors. In our particular framework we introduce a similar type of spinors with respect to our connection $\bar{\nabla}$.

Definition 2 *A spinor* $s \in \Gamma(S)$ *which satisfies the equation*

$$\bar{\nabla}_X s + \frac{\lambda}{q} X \cdot s = 0,$$

for any $X \in \Gamma(Q)$ *is called* transverse Killing spinor associated with the connection $\bar{\nabla}$ *or* $\tau-$Killing spinor.

Remark 6 *As in the standard setting, it is easy to see that a basic harmonic spinor is a transverse Killing spinor and each transverse Killing spinor is an eigenspinor.*

The concept of Killing spinors can be extended to *twistor spinors* in a classical manner [3, 11].

Definition 3 *We denote by* $\tau-$ twistor spinors (twistor spinors with respect to the modified connection) *a basic spinor satisfying the following equation*

$$\bar{\nabla}_X s + \frac{1}{q} X \cdot D_b s = 0. \tag{4}$$

Remark 7 *We will see in the next sections that these definitions extend previous basic spinors existing in the particular case of Riemannian foliations with basic-harmonic mean curvature [10, 17, 18]. As the linear connection employed to define* τ*-Killing spinors in not metric, it will be interesting to point out that they do not have constant length, unlike the previous definition.*

Now, as we have defined all necessary transverse geometric objects, in the final part of this section we shortly present some results that we employ in order to study spectral properties of basic Dirac operators. It turns out that, the most convenient setting is represented by the case of Riemannian foliations with basic-harmonic mean curvature, that is $d_b k = 0$, $\delta_b k = 0$. Then, in order to obtain results in the general case, we need a sequence of metric changes that leave the transverse metric on the normal bundle intact.

A first relevant result in this direction is due to Domínguez.

Theorem 1 *[7] The bundle-like metric can be transformed (leaving the transverse metric on the normal bundle intact) such that the new bundle-like metric have basic mean curvature.*

The above metric transformation is based on a change of the transversal sub-bundle Q, and a conformal change of the leafwise metric, this being in fact the fundamental metric changes needed in order to study the basic component of the mean curvature [1]. The composition of these metric changes ensure the vanishing of the orthogonal part k_o of the mean curvature form k, while the basic part k_b holds.

Secondly, we have the following result obtained by Mason.

Theorem 2 *[24] Furthermore, the above bundle-like metric can be transformed (leaving the transverse metric on the normal bundle intact) into a metric with basic-harmonic mean curvature.*

This metric change is in fact a conformal change of the leafwise metric which use the theory of stochastic flows.

Finally, an important spectral rigidity result is due to Habib and Richardson.

Theorem 3 *[13] The spectrum $\sigma(D_b)$ is invariant with respect to any metric change that leaves the transverse metric on the normal bundle intact.*

As a consequence, we can study first of all the spectrum of basic Dirac operator in the most convenient framework of Riemannian foliations with basic-harmonic mean curvature, then pull back the results in the initial general case using the spectral rigidity result [13]. For the case $q \geq 2$ the authors obtain that the lower bound estimate for an eigenvalue λ of D_b

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1}\mathrm{Scal}_0^{\nabla},$$

where $\mathrm{Scal}_0^{\nabla} := \min_{x\in M} \mathrm{Scal}_x^{\nabla}$, as an extension of [17].

While the above method is very useful when studying the eigenvalues of D_b, in general the corresponding eigenspinors not invariant with respect to all the above changes of the metric.

3 Basic Dirac operators and transverse $\tau-$Killing spinors

In the framework of closed Riemannian manifold with spin structure, the interplay between Dirac operator and Killing spinors provide some remarkable and very interesting results [3, 11]. In this section we study this interplay between basic Dirac operator and transverse $\tau-$Killing spinors on Riemannian foliations, obtaining corresponding results in our specific setting, generalizing also known results from basic-harmonic Riemannian foliations [17, 18].

Definition 4 *For any basic function f we define the linear connections ∇^f_X, and $\bar{\nabla}^f_X$, defined by the relations*

$$\nabla^f_X := \nabla_X + fX\cdot,$$
$$\bar{\nabla}^f_X := \bar{\nabla}_X + fX\cdot.$$

for any $X \in \Gamma(Q)$.

Proposition 1 *Considering now the corresponding curvature operators, we have the equality $\bar{R}^f_{X,Y} = R^f_{X,Y}$ for $X,\ Y \in \Gamma(Q)$.*

Proof: Using the standard definition of the curvature operator, we start with the relation

$$\bar{R}^f_{X,Y} = \bar{\nabla}^f_X \bar{\nabla}^f_Y - \bar{\nabla}^f_Y \bar{\nabla}^f_X - \bar{\nabla}^f_{[X,Y]}. \tag{5}$$

Furthermore, we get

$$\begin{aligned}
\bar{\nabla}^f_X \bar{\nabla}^f_Y &= \left(\nabla^f_X - \frac{1}{2}\langle X, \tau\rangle\right)\left(\nabla^f_Y - \frac{1}{2}\langle Y, \tau\rangle\right) \\
&= \nabla^f_X \nabla^f_Y - \frac{1}{2}\langle \nabla_X Y, \tau\rangle - \frac{1}{2}\langle Y, \nabla_X \tau\rangle \\
&\quad -\frac{1}{2}\langle Y, \tau\rangle \nabla_X - \frac{1}{2} f\langle Y, \tau\rangle X - \frac{1}{2}\langle X, \tau\rangle \nabla_Y \\
&\quad -\frac{1}{2} f\langle X, \tau\rangle Y + \frac{1}{4}\langle X, \tau\rangle \langle Y, \tau\rangle .
\end{aligned}$$

We also have the corresponding relation for the second term of 5; for the third term we have

$$\bar{\nabla}^f_{[X,Y]} = \nabla^f_{[X,Y]} - \frac{1}{2}\langle [X,Y], \tau\rangle .$$

Now, let us emphasize that k_b is a closed 1-form [1], so

$$\langle Y, \nabla_X \tau\rangle = \langle X, \nabla_Y \tau\rangle .$$

Also, the Bott connection is torsion-free, so we get

$$\langle \nabla_X Y - \nabla_X Y - [X,Y], \tau\rangle = 0.$$

Summing up, we observe that all terms containing τ vanish and we obtain the above relation. $\square$

Remark 8 *For $f \equiv 0$ we derive that $\bar{R}_{X,Y} = R_{X,Y}$.*

Proposition 2 *If we assume the existence of a basic spinor $s_1 \in \Gamma(S)$ which verifies the equation*

$$\bar{\nabla}_X s_1 + \frac{f}{q} X \cdot s_1 = 0, \tag{6}$$

for any $X \in \Gamma(Q)$, f being a basic function, then f is constant $f \equiv \lambda_1$, (as a consequence the spinor is transverse $\tau-$Killing). If the codimension $q \geq 2$, then the foliation is transversally Einstein, the transverse scalar curvature Scal^{∇} *is a positive constant function* $\mathrm{Scal}^{\nabla} \equiv Scal_0^{\nabla} > 0$*, and*

$$\lambda_1^2 = \frac{1}{4}\frac{q}{(q-1)}\mathrm{Scal}_0^{\nabla}.$$

Proof: From (6) we have

$$\begin{aligned} 0 &= \sum_i e_i \cdot \bar{R}^f_{X,e_i} s_1 \\ &= \sum_i e_i \cdot R^f_{X,e_i} s_1. \end{aligned}$$

Then, arguing as in [17] (for the classical framework of a closed Riemannian see [15]) we obtain

$$-\frac{1}{2}\sum_i R_{X,e_i} e_i \cdot s_1 - qX(f)s_1 - \mathrm{grad}^{\nabla}(f)X \cdot s_1 + f^2 2(q-1)X \cdot s_1 = 0,$$

where grad^{∇} is the *transversal gradient* of the basic function f. Now, the conclusion comes from [17]. $\square$

So, as a remark, the above fundamental properties of Killing spinor still hold for our particular definition.

Theorem 4 *An arbitrary Riemannian foliation admits a transverse $\tau-$Killing spinor if and only if this spinor is the eigenspinor of the basic Dirac operator for which the lower bound estimate of the spectrum is realized.*

Proof: If we assume the existence of a transverse Killing spinor s_1 associated with $\bar{\nabla}$, then

$$\begin{aligned} D_b s_1 &= \sum_i e_i \cdot \bar{\nabla}_{e_i} s_1 \\ &= \left(\sum_i e_i \cdot \frac{1}{q}\left(-\frac{1}{4}\frac{q}{q-1}\mathrm{Scal}_0^{\nabla}\right) e_i\right) s_1 \\ &= -\frac{1}{4}\frac{q}{q-1}\mathrm{Scal}_0^{\nabla} s_1, \end{aligned}$$

so the lower bound estimate is attained, in accordance with [17, 13].

For the converse statement let us consider a deformation of the metric as in [7] which turns the mean curvature form into a basic 1-form. If the lower bound estimate of the spectrum is realized for the initial metric, using the spectral rigidity result [13], the lower bound estimate will be attainted for the case of basic mean curvature form. In this particular setting we have the Lichnerowicz type formula [30]

$$\|D_b s\|^2 = \left\|\bar{\nabla} s\right\|^2 + \frac{1}{4}\int\limits_M \mathrm{Scal}^{\nabla} |s|^2 , \tag{7}$$

for $s \in \Gamma_b(S)$, where $|s|^2 = (s \mid s)$, $\|\cdot\|$ being the L^2 norm associated with the hermitian structure.

In what follows, let us consider the connection

$$\bar{\nabla}_X^{\frac{\lambda_1}{q}} := \bar{\nabla}_X + \frac{\lambda_1}{q} X \cdot .$$

when λ_1 is the eigenvalue for which the lower bound is attained, s_1 being the corresponding eigenspinor. Then, following closely the original approach due to Friedrich [8], using (7), after calculations we get [30]

$$\int\limits_M \left| D_b s_1 - \frac{\lambda_1}{q} s_1 \right|^2 = \int\limits_M \sum_i \left| \bar{\nabla}_{e_i}^{\frac{\lambda_1}{q}} s_1 \right|^2 + \frac{1}{4}\int\limits_M \mathrm{Scal}^{\nabla} |s_1|^2 + \int\limits_M (1-q)\frac{\lambda_1^2}{q^2} |s_1|^2 ,$$

and, consequently [30]

$$\int\limits_M \left(\frac{q-1}{q}\lambda_1^2 - \frac{1}{4} Scal^{\nabla} \right) |s_1|^2 = \int\limits_M \sum_i \left| \bar{\nabla}_{e_i}^{\frac{\lambda_1}{q}} s_1 \right|^2 .$$

From here, it turns out that

$$\int\limits_M \sum_i \left| \bar{\nabla}_{e_i}^{\frac{\lambda_1}{q}} s_1 \right|^2 = 0 \text{ and } \lambda_1^2 = \frac{1}{4}\frac{q}{q-1}\mathrm{Scal}_x^{\nabla} ,$$

so

$$\bar{\nabla}_X s + \frac{\lambda_1}{q} X \cdot s = 0,$$

for any $X \in \Gamma(Q)$, i.e. the spinor s is a transverse $\tau-$Killing spinor.

As we noticed above, the metric change described in [7] leaves the transverse metric and the basic part k_b of the mean curvature intact, so the action of the modified connection $\bar{\nabla}$ on $\Gamma_b(E)$ does not changes. On the other side the Clifford multiplications, related to the transverse metric always agree. As a consequence, the spinor s will be a transverse Killing spinor associated with $\bar{\nabla}$ for the initial metric tensor, and we extend our result to arbitrary Riemannian foliations using [7]. □

Proposition 3 *If a Riemannian foliation admits a transverse Killing spinor with respect to the connection $\bar{\nabla}$, then the foliation is taut.*

Proof: Assuming that the Riemannian foliations admit a transverse Killing spinor with respect to the connection $\bar{\nabla}$, we obtain that there is an eigenspinor s_1 for which the lower bound estimate of the spectrum of the basic Dirac operator

is attained; from the Proposition 2 we also obtain that in the case $q \geq 2$ the foliation should be transversally Einstein, and the transverse scalar curvature should be positive and constant.

Using now once again the rigidity of the spectrum $\sigma(D_b)$, we obtain that using a sequence of metric changes that hold the transversal part of the metric we end up with a Riemannian foliation with a basic-harmonic mean curvature for which the above lower bound is also realized and the square of the first eigenvalue is again $\frac{1}{4}\frac{q}{q-1}\mathrm{Scal}_0^{\nabla}$. From [17] we get that with respect to the deformed metric the foliation should be minimal, i.e. $k \equiv 0$.

As a consequence, by the above sequences of metric changes we obtain a Riemannian foliation with minimal leaves; as this is the standard definition for a taut foliation (see e. g. [1]), the conclusion follows. □

Remark 9 *For foliations with basic-harmonic mean curvature the existence of a transverse Killing spinor is restricted by the condition $k \equiv 0$; as a result $\bar{\nabla} \equiv \nabla$, and a spinors is transversally Killing with respect to $\bar{\nabla}$ and ∇ in the same time. Consequently, the above results are natural generalization of [17, Theorem 5.3] and [18, Corollary 4.5].*

4 Transverse $\tau-$ twistor spinors

Within this section, in the same framework of Riemannian foliations with non-necessarily basic-harmonic mean curvature, using our previous method we study the main properties of the transverse $\tau-$ twistor spinors introduced in Section 2.

First of all we see that this concept is in fact an extension of [18]. Indeed, by [18, Theorem 3.2], the twistor spinors exist only on minimal foliations. As for minimal foliations we have $\tau = 0$, we see that the above definition agrees with [18] for this particular class of Riemannian foliations; as the harmonic spinor considered in the Section 2 is also a twistor spinor defined on a taut, non-minimal foliation, the above definition is in fact an extension of [18]; however, the fact that the $\tau-$twistor spinors exist only on taut foliation (as well as Killing spinors) does not seem to be a direct consequence.

The following relations from [17] will be useful in our further considerations

$$\sum_i e_i \cdot R_{X,e_i} s = -\frac{1}{2}\mathrm{Ric}^{\nabla}(X) \cdot s,$$
$$\sum_i e_i \cdot \mathrm{Ric}^{\nabla}(X) = -\mathrm{Scal}^{\nabla} s.$$

In the calculations below we use the above relations, the fact that Bott connection has vanishing torsion and is compatible with the modified connection, as well as the fact that the basic spinors are parallel on the leafwise directions.

$$\frac{1}{2}\mathrm{Ric}^{\nabla}(X) \cdot s \quad = \quad -\sum_i e_i \cdot R_{X,e_j} s$$

$$
\begin{aligned}
&= -\sum_i e_i \cdot \bar{R}_{X,e_j} s \\
&= -\sum_i e_i \cdot \left(\bar{\nabla}_X \bar{\nabla}_{e_i} - \bar{\nabla}_{e_i} \bar{\nabla}_X - \bar{\nabla}_{\pi_Q([X,e_i])}\right) s \\
&= -\sum_i e_i \cdot \bar{\nabla}_X \left(-\frac{1}{q} e_i \cdot D_b s\right) - \bar{\nabla}_{e_i} \left(-\frac{1}{q} X \cdot D_b s\right) \\
&\quad -\pi_Q([X,e_i]) \cdot D_b s \\
&= -\sum_i e_i \cdot \left(-\frac{1}{q} e_i \cdot \bar{\nabla}_X D_b s + \frac{1}{q} \bar{\nabla}_X D_b s\right) \\
&\quad +\frac{1}{q} \sum_i e_i \cdot \left(\nabla_X e_i - \nabla_{e_i} X - \pi_Q([X,e_i])\right) \cdot D_b s.
\end{aligned}
$$

Now, as the transverse part of the torsion tensor associated to the Bott connection vanishes, using standard computation (see e. g. [11]), we get

$$
\bar{\nabla}_X D_b s = -\frac{q}{2\,(q-2)} \mathrm{Ric}^{\nabla}(X) \cdot s - \frac{1}{q-2} X \cdot D_b^2 s.
$$

From here

$$
D_b^2 s = \frac{q}{4(q-1)} \mathrm{Scal}^{\nabla} s, \tag{8}
$$

and we obtain the corresponding upgrading

$$
\bar{\nabla}_X D_b s = \frac{q}{(q-2)} \left(\mathrm{Ric}^{\nabla}(X) \cdot s + \frac{4}{q-1} \mathrm{Scal}^{\nabla} X \cdot s\right). \tag{9}
$$

Remark 10 *The relations (8) and (9) represent generalizations of some results from [18]. Using these calculus properties for twistor spinors we can easily obtain standard results regarding twistor spinors, as well as relations between twistor and Killing spinors, just arguing similar to [11, 8].*

In the last part of the paper we prove an interesting property of the zeros of twistor spinors.

First of all let us consider the bundle $E := S \bigoplus S$, endowed with the connection

$$
\nabla^E_X \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} := \begin{pmatrix} \bar{\nabla}_{\pi_Q(X)} & \frac{1}{q}\pi_Q(X)\cdot \\ \frac{q}{(q-2)} \left(\mathrm{Ric}^{\nabla}(X) + \frac{4}{q-1}\mathrm{Scal}^{\nabla} X\right) \cdot & \bar{\nabla}_{\pi_Q(X)} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},
$$

for any s_1, $s_2 \in \Gamma(S)$, as a generalization of the basic-harmonic case [18] (for the classical case see e.g. [3]). It is easy to see that if s is a twistor spinor, then the smooth section $\begin{pmatrix} s \\ D_b s \end{pmatrix}$ of E is in fact parallel with respect to ∇^E, as a consequence of (4), (9) and the definition of ∇^E. Considering arguments similar to [3] for the transverse directions, as the spinors s and $D_b s$ are basic

spinors, parallel along the leaves, defined by (1), we see that if the manifold M is connected and at a point $x \in M$ we have $s_x = (D_b s)_x = 0$, then $s \equiv 0$ all over the compact manifold M.

For any basic function f we define the *basic Hessian* associated to the connection ∇

$$\mathrm{Hess}^{\nabla}(f)(X,Y) := X\left(Y\left(f\right)\right) - \nabla_X Y\left(f\right) \tag{10}$$

for any X, $Y \in \Gamma(Q)$.

Remark 11 *It T is a local transversal manifold, as basic geometric objects in our framework can be locally projected on T, it is easy to see that* Hess^{∇} *is just the standard Hessian on the transversal manifold.*

We are now able to prove the corresponding version of a classical property of the zeros of a twistor spinor [3, 11].

Proposition 4 *On a connected Riemannian foliation endowed with a nontrivial transverse τ-twistor spinor s, the leaves where s vanishes are isolated on the quotient set $M_{/\mathcal{F}}$.*

Proof: Let us assume s is a nontrivial basic twistor spinor such that $s_x = 0$ at $x \in M$. In the following we calculate $\mathrm{Hess}^{\nabla}_x(|s|^2)$.

We investigate the two components of the basic Hessian defined by (10).

$$\begin{aligned} X\left(Y\left(|s|^2\right)\right) &= X\left(2\,Re\left(\nabla_Y s \mid s\right)\right) \\ &= 2\,Re\left(\nabla_X \nabla_Y s \mid s\right) + 2\,Re\left(\nabla_Y s \mid \nabla_X s\right), \end{aligned}$$

$$\nabla_X Y\left(|s|^2\right) = 2\,Re\left(\nabla_{\nabla_X Y} s \mid s\right).$$

As $s_x = 0$, the only term we need to study is $Re\left(\nabla_Y s \mid \nabla_X s\right)$.

We now apply (4) and (3) and obtain that

$$\begin{aligned} \left(\nabla_Y s \mid \nabla_X s\right) &= \left(\bar{\nabla}_Y s \mid \bar{\nabla}_X s\right) + \frac{1}{2}\left\langle Y,\tau\right\rangle\left(s \mid \nabla_X s\right) \\ &\quad + \frac{1}{2}\left\langle X,\tau\right\rangle\left(\nabla_Y s \mid s\right) + \frac{1}{4}\left\langle Y,\tau\right\rangle\left\langle X,\tau\right\rangle \\ &= \frac{1}{q^2}\left\langle X\cdot D_b s, Y\cdot D_b s\right\rangle + \frac{1}{2}\left\langle Y,\tau\right\rangle\left(s \mid \nabla_X s\right) \\ &\quad + \frac{1}{2}\left\langle X,\tau\right\rangle\left(\nabla_Y s \mid s\right) + \frac{1}{4}\left\langle Y,\tau\right\rangle\left\langle X,\tau\right\rangle |s|^2. \end{aligned}$$

From [3, p. 15] we get furthermore that

$$\left\langle X\cdot D_b s, Y\cdot D_b s\right\rangle = \left\langle X,Y\right\rangle |D_b s|^2,$$

so at the point x, as $s_x = 0$, we obtain

$$\begin{aligned} \mathrm{Hess}^{\nabla}_x(|s|^2)(X_x,Y_x) &= Re\left(\left(\nabla_Y s\right)_x \mid \left(\nabla_X s\right)_x\right) \\ &= \frac{1}{q^2}\left\langle X_x,Y_x\right\rangle |D_b s|^2_x. \end{aligned}$$

As $(D_b s)_x \neq 0$ (otherwise, in accordance with the above considerations the twistor spinor s would vanish everywhere), we get that the basic Hessian of the local function obtained on the local transversal manifold T by projecting the basic function $|s|^2$ is positive defined at x; as the basic functions are constant along the leaves of the foliation, the conclusion follows. □

5 Some physical considerations

The Dirac operators in the presence of Riemannian foliations have attracted much attention in physics. We do not intend to give any kind of extensive introduction in spin geometry, but some motivating examples that may lead to possible applications are of interest.

In the last time Sasakian manifolds, as an odd-dimensional cousin of Kähler manifolds, have become of high interest in connection with many modern studies in physics. One of their principal applications in physics has been in higher-dimensional supergravity, string theory and M-theory where they can provide backgrounds for reduction to lower-dimensional spacetimes. AdS/CFT conjecture [21] relates quantum gravity, in certain backgrounds, to ordinary quantum field theory without gravity. In particular the AdS/CFT correspondence relates Sasaki-Einstein geometry, in dimensions five and seven, to superconformal field theory in dimensions four and three, respectively.

The foliation generated by the Reeb vector field ξ has a transverse Kähler structure. If the orbits of ξ are closed, the Sasakian structure is called *quasi-regular*. The Reeb field generates a locally free S^1-action such that the leaf space is an orbifold and the transverse Kähler structure projects to it. There are examples of Sasakian structures which are not quasi-regular [9]. In the opposite case, if the orbits of ξ do not all close, the Sasakian structure is said to be *irregular*.

The properties of Sasaki-Einstein spaces can be obtained from an alternative definition of Sasaki-Einstein manifold connected with the existence of a Killing spinor [2]. The geometric features of generic supergravity solutions with unbroken supersymmetry and fluxes, so the relation between Killing spinor and geometry that admits such a spinor needs to be further elucidated [19]. On the other side, as pointed out in [11], the Killing spinors are highly relevant for the investigation of supersymetric models for string theory in dimension 10. From this point of view we hope that our results concerning transverse Killing spinors would be helpful for the investigation of this geometrical objects in the particular framework represented by Riemannian foliations.

Another interesting example is represented by the Euclidean Taub-Newman-Unti-Tamburino (Taub-NUT) space which appears in various problems. Hawking [14] has suggested that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the Yang-Mills instantons. Also this metric is the space part of the line element of the Kaluza-Klein monopole.

Iwai and Katayama [16] generalized the Taub-NUT metrics in the following way. Let us consider a metric $\bar{g}$ on an open interval U in $(0, +\infty)$ and a family

of Berger metrics $\hat{g}(r)$ on S^3 indexed by U. Then the twisted product metrics $g = \bar{g} + \hat{g}(r)$ on the annulus $U \times S^3 \subset \mathbb{R}^4 \backslash \{0\}$ is called a generalized Taub-NUT metric.

The Taub-NUT metrics has been drawing wide interest. In particular, from the viewpoint of dynamical systems, the symmetry of the dynamical system associated with that metric is similar to that for the Coulomb/Kepler problem. The four-dimensional problem is reduced to an S^1 action when the associated momentum mapping takes nonzero fixed values. If the original Hamiltonian system admits a symmetry group that is commutative with the group used for the reduction, the reduced Hamiltonian system admits the same symmetry group [23]. In the case of the Taub-NUT metrics the reduced Hamiltonian system is the three-dimensional Kepler problem along with a centrifugal potential and Dirac's monopole field.

The importance of anomalous Ward identities in particle physics is well known. The anomalous divergence of the axial vector current in a background gravitational field is directly related to the index theorem. Namely, the axial anomaly is interpreted as the index of the chiral Dirac operator. The difference between the number of null states of positive and of negative chirality on a ball or annular domain, may become nonzero for suitable choices of the parameters of the metric and of the domain when one imposes the Atiyah-Patodi-Singer spectral condition at the boundary. In the case of the standard Taub-NUT space, which is hyperKähler and therefore scalar-flat, it can be proved that there are no harmonic L^2 spinors using the Lichnerowicz identity and the infiniteness of the volume [26]. Moreover there do not exist L^2 harmonic spinors on $\mathbb{R}^4$ for the generalized Taub-NUT metrics. In particular, the L^2 index of the Dirac operator vanishes [25].

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Adrian Mihai IONESCU
Politehnica University of Bucharest,
Department of Mathematics,
Splaiul Independenţei, Nr. 313, Sector 6,
Bucureşti 060042-ROMANIA

E-mail: aionescu@math.pub.ro

Vladimir SLESAR
University of Craiova,
Department of Mathematics,
Str. Al.I. Cuza, Nr. 13,
Craiova 200585-ROMANIA
E-mail: vlslesar@central.ucv.ro

Mihai VISINESCU
National Institute for Physics and Nuclear Engineering,
Department of Theoretical Physics,
P.O.Box M.G.-6, Magurele, Bucharest-ROMANIA
E-mail: mvisin@theory.nipne.ro

Gabriel Eduard VÎLCU[1,2]
[1]University of Bucharest, Faculty of Mathematics and Computer Science,
Research Center in Geometry, Topology and Algebra,
Str. Academiei, Nr. 14, Sector 1,
Bucureşti 70109-ROMANIA
E-mail: gvilcu@gta.math.unibuc.ro
[2]Petroleum-Gas University of Ploieşti,
Department of Mathematical Modelling, Economic Analysis and Statistics,
Bulevardul Bucureşti, Nr. 39
Ploieşti 100680-ROMANIA
E-mail: gvilcu@upg-ploiesti.ro